Generalised Pontryagin Construction for CW Complexes

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Abstract

The well-known construction that was first introduced by L. S. Pontryagin in [1] provides a geometric characterisation of homotopy groups of spheres, by equating them with cobordism groups of framed submanifolds of Euclidean space. The theory of cobordisms was then vastly expanded by R. Thom in [3], in which he defines the oriented and unoriented cobordism groups and relates them to the homotopy groups of some specially constructed spaces. Here we will answer the question: what sort of "cobordism group" corresponds to the homotopy groups of an arbitrary finite CW complex X? We introduce the notion of an $\widetilde{X}$-manifold, which can be thought of as a particular species of stratified space equipped with a normal framing, whose singular structure is governed by the structure of the complex X. We then prove that the homotopy groups of the CW complex X are isomorphic to the cobordism groups of $\widetilde{X}$-manifolds, and demonstrate some examples of this new geometric presentation.

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1 Introduction

The classical Pontryagin construction provides an isomorphism between the homotopy groups of spheres and the cobordism groups of framed submanifolds of Euclidean space, good accounts can be found in e.g. [2] and [4]. Starting with cobordism groups of manifolds with some other structure such as orientation (or none at all), Thom found spaces whose homotopy groups are related by an analogous construction. One can also turn the question the other way, and ask: what sort of cobordism theory arises out of considering the homotopy groups of a particular space? The objective of this paper is to answer this question when the space is a finite CW complex X.

As it turns out, the objects of the resulting cobordism group are no longer manifolds, but *stratified spaces* with a singular structure that arises out of the cellular structure of X. These spaces are endowed with normal framing information on their strata, also in a way that is restricted by X. Although it is likely possible to develop this theory using the language of stratified spaces and normal framings, and imposing additional compatibility conditions, we will instead take the route of defining the spaces directly in a way that suits our needs. We will define objects called $\widetilde{X}$*-manifolds*: very roughly speaking, these are subspaces of Euclidean space that are locally endowed with tubular neighbourhoods that fibre over regions of the "dual complex" $\widetilde{X}$. A possibility for further research would be to take the other route and show equivalence of this formulation with one that uses stratified spaces in the sense of Thom or Whitney.

After stating the necessary definitions and going through the construction, we showcase the geometric appeal of this approach by working through examples of low-dimensional homotopy groups when X is S^n, $\mathbb{R}P^2$, $\Sigma\mathbb{R}P^2$, and $S^n \vee S^m$.

2 $\widetilde{X}$-manifolds and their cobordisms

2.1 Preliminaries

For a space Y, the *cone on* Y is the space $\mathrm{Cone}(Y) := ((Y \times I) \coprod *) / \sim$ where $\sim$ is the relation given by $(y, 1) \sim *$ for all $y \in Y$. Note that in particular, $\mathrm{Cone}(\emptyset) = *$. In the special case when $Y \subseteq S^n = \partial D^{n+1}$, we view $\mathrm{Cone}(Y)$ as a subset of D^{n+1}, with the cone point at the centre of the $(n+1)$-disk, by identifying $\mathrm{Cone}(S^n) \cong D^{n+1}$.

We start with a finite CW-complex X with skeleta $X^0 \subset X^1 \subset \cdots \subset X^N$, characteristic maps $g_{ij} : D^i \to X$ for each $0 \leq i \leq N$ and $1 \leq j \leq n_i$, where n_i is the number of i-cells in X, with a selected 0-cell $x_0 \in X^0$ which will serve as the basepoint. We shall write $\mathring{D}^i_{(j)}$ for $g_{ij}(\mathrm{Int}(D^i))$, the image of the interior of the jth i-cell in X, and c_{ij} for the centre point of $\mathring{D}^i_j$.

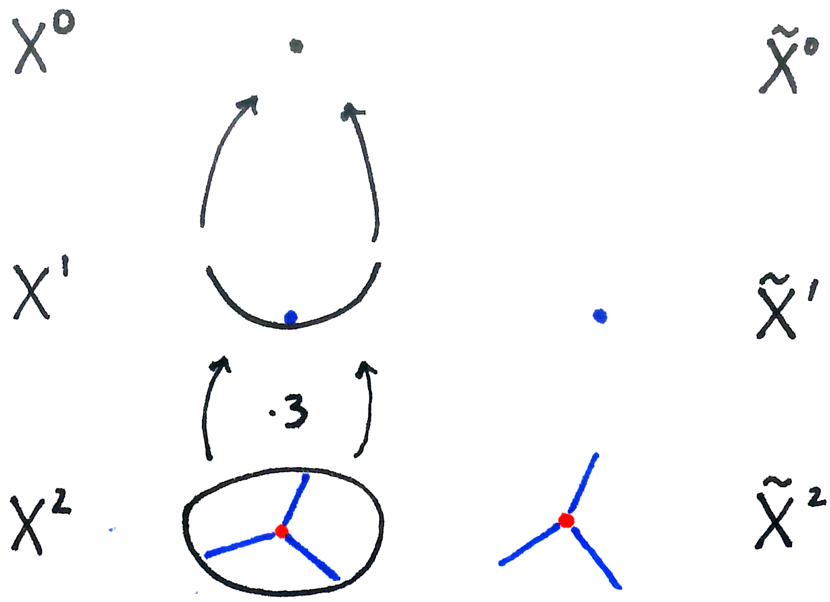


Figure 1: Construction of $\widetilde{X}$ when $X = e_0 \cup e_1 \cup_{\cdot 3} e_2$

2.2 Construction of $\widetilde{X}$

We construct the filtered space $\widetilde{X}^0 \subset \widetilde{X}^1 \subset \cdots \subset \widetilde{X}^N =: \widetilde{X}$ as follows:

- $\widetilde{X}^0$ consists of all 0-cells of X except for x_0
- $\widetilde{X}^i = \widetilde{X}^{i-1} \cup \bigcup_{j=1}^{n_i} C^i_j$ where

$$C^i_j = g_{ij}(\mathrm{Cone}(g_{ij}^{-1}(\widetilde{X}^{i-1}))) \cap \mathring{D}^i_{(j)}$$

  Inductively $\widetilde{X}^i \hookrightarrow X^i$ since $g_{ij}\left(\partial D^i\right) \subset X^{i-1}$ and so $g_{ij}^{-1}(\widetilde{X}^{k-1}) \subset \partial D^i$. We have the inclusion $C^i_j \hookrightarrow \mathring{D}^i_{(j)}$ where the cone point is identified with c_{ij}.

n.b. that $\widetilde{X}$ is a closed subset of X.

1 Example. If X is the CW decomposition of S^n with one n-cell, then $\widetilde{X} = *$. As we shall see, this recovers the classical construction for homotopy groups of S^n.

2 Example. Consider X constructed by attaching a 2-cell to a circle by the $\cdot 3$ map. $\widetilde{X}$ looks like a tripod in this case (see figure 1).

3 Example. Let X be a cell complex with one cell in dimensions 1, 2, and 4: the 2-cell is glued by the $\cdot 2$ map, and the 4-cell is glued along $q \circ \mathfrak{h}$, where $q : S^2 \to \mathbb{R}P^2$ is the double cover and $\mathfrak{h} : S^3 \to S^2$ is the Hopf map. $\widetilde{X}$ looks like a cone with a line drawn on it that passes through the cone point (see figure 2).

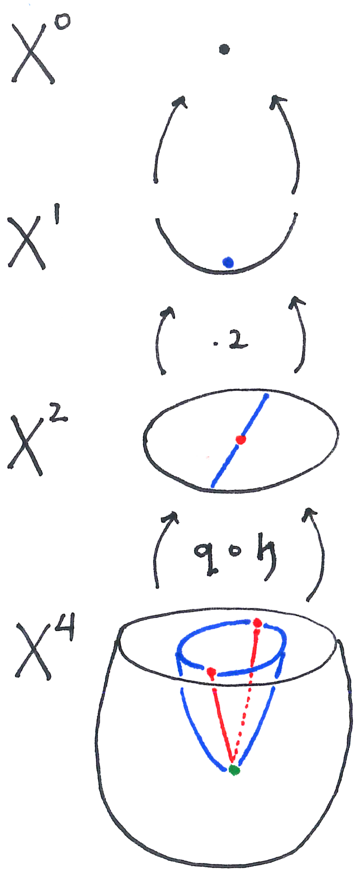


Figure 2: Construction of $\widetilde{X}$ when X is the complex described in example 3

We need to define a subdomain of X on which an iterated retraction can be defined. Let $X^i_{\text{punc}} := X^i \setminus \bigcup_{j=1}^{n_i} \{c_{ij}\}$, $\rho_i : X^i_{\text{punc}} \to X^{i-1}$ be the retraction consisting of retractions on each punctured i-cell. Next, define:

$$X^i_{0\,\text{punc}} := X^i$$
$$X^i_{k\,\text{punc}} := \rho_i^{-1}(X^{i-1}_{(k-1)\,\text{punc}})$$

for $i \geq k \geq 1$, and define maps:

$$\rho^0_{i,0} := \mathrm{id}_{X^i} \qquad : X^i \to X^i$$
$$\rho^n_{i,k} := \rho^{n-1}_{i-1,k-1} \circ \rho_i\big|_{X^i_{k\,\text{punc}}} : X^i_{k\,\text{punc}} \to X^{i-n}_{(k-n)\,\text{punc}}$$

for $1 \leq n \leq k$. By $\rho_i^{(t)}$ we will mean the deformation retraction defined linearly on each cell, such that $\rho_i^{(0)} = \mathrm{id}$ and $\rho_i^{(1)} = \rho_i$.

2.3 The main definition

Let $\mathring{D}^i_j(\epsilon) := g_{ij}(N_\epsilon(0)) \subseteq \mathring{D}^i_j$, where $N_\epsilon(0)$ is the ball of radius ϵ around $0 \in D^i$, and let $C^i_j(\epsilon) := C^i_j \cap \mathring{D}^i_j(\epsilon)$ for $0 < \epsilon \leq 1$.

4 Definition. A *$\widetilde{X}$-manifold in $\mathbb{R}^m$* is a compact subset $M \subset \mathbb{R}^m$ with an *$\widetilde{X}$-atlas*: a collection $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ such that the $\mathcal{U}_\alpha$ cover M and where each $\mathcal{U}_\alpha$ is open in $\mathbb{R}^m$, and each

$$\phi_\alpha : \mathcal{U}_\alpha \xrightarrow{\sim} \mathring{D}^{i_\alpha}_{j_\alpha}(\epsilon) \times \mathbb{R}^{m-i_\alpha}$$

is a diffeomorphism for some $i_\alpha \leq m$ and $1 \leq j_\alpha \leq n_i$ that restricts to a homeomorphism

$$\phi_\alpha\big|_{\mathcal{U}_\alpha \cap M} : \mathcal{U}_\alpha \cap M \xrightarrow{\sim} C^{i_\alpha}_{j_\alpha}(\epsilon) \times \mathbb{R}^{m-i_\alpha}$$

and such that

1. Each $\mathcal{U}_\alpha$ is bounded, with smooth boundary
2. Each ϕ_α extends uniquely to a map on $\bar{\mathcal{U}}$.
3. For any two charts $\mathcal{U}$ and $\mathcal{V}$, $\partial\mathcal{U} \pitchfork \partial\mathcal{V}$.
4. Any two intersecting charts are immediate, in the following sense:

We write $\widetilde{\phi}$ for the projected chart $\mathcal{U} \xrightarrow{\phi} \mathring{D}^i_j(\epsilon) \times \mathbb{R}^{m-i} \to \mathring{D}^i_j(\epsilon)$.

5 Definition. Two intersecting charts $\phi : \mathcal{U} \xrightarrow{\sim} \mathring{D}^i_j(\epsilon_1) \times \mathbb{R}^{m-i}$ and $\psi : \mathcal{V} \xrightarrow{\sim} \mathring{D}^k_l(\epsilon_2) \times \mathbb{R}^{m-k}$ with $i \geq k$ are *immediate* if $\widetilde{\phi}(\mathcal{U} \cap \mathcal{V}) \subseteq X^i_{(i-k)\,\mathrm{punc}}$ and the following diagram commutes:

$$\begin{tikzcd}
\mathcal{U} \cap \mathcal{V} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\widetilde{\psi}"] & \widetilde{\phi}(\mathcal{U} \cap \mathcal{V}) \arrow[d, "\rho^{i-k}_i"] \\
\widetilde{\psi}(\mathcal{U} \cap \mathcal{V}) \arrow[r, hook] & X^k
\end{tikzcd}$$

n.b. that (an embedding of) $\widetilde{X}$ itself need not be an $\widetilde{X}$-manifold.

6 Example. When X is the CW-decomposition of S^n with one n-cell, the immediacy condition holds vacuously and an $\widetilde{X}$-manifold is just a $(m-n)$-submanifold of $\mathbb{R}^m$ with a tubular neighbourhood, or equivalently, a normally framed submanifold of $\mathbb{R}^m$.

Remark. The purpose of the immediacy conditions is to require that the features inherited from $\widetilde{X}$ that are close together in M must correspond to adjacent features in $\widetilde{X}$. In §4, this will allow us to glue together charts of an $\widetilde{X}$-atlas to construct a map into X.

Remark. The notion of $\widetilde{X}$-manifold is very much dependent on the particular choice of cell structure on X, i.e. it is not a homotopy invariant of X (or at least not in any obvious way). In §5, we will explicitly demonstrate how different cell structures can give rise to different presentations.

(a) $X = e_0 \cup e_1 \cup_{\cdot 1} e_2 \cong D^2$

(b) $X = e_0 \cup e_1 \cup_{\cdot 2} e_2 \cong \mathbb{R}P^2$

Figure 3: Some examples of $\widetilde{X}$-manifolds in $\mathbb{R}^2$. We draw "hairs" purely as an illustrative tool to evoke the right charts. In §5, we will discuss how these can be used more formally to characterise the $\widetilde{X}$-manifold.

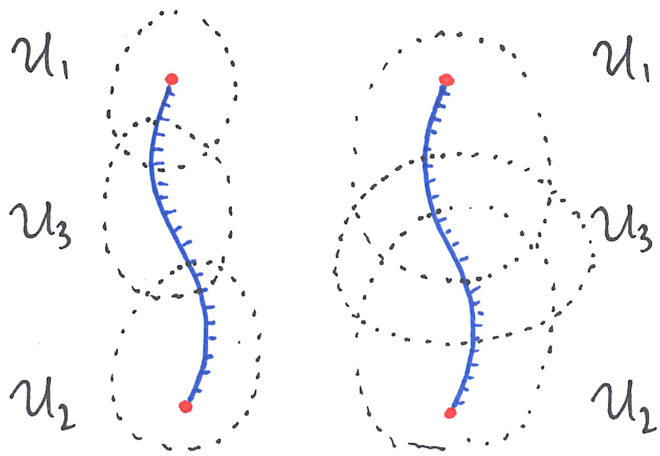


Figure 4: The first is an $\widetilde{X}$-atlas for D^2. The second isn't, since $\mathcal{U}_1$ and $\mathcal{U}_2$ are not immediate.

7 Definition. An $\widetilde{X}$*-manifold with boundary in* $\mathbb{R}^m$ is the same as above except that for some α we may have instead:

$$\phi_\alpha : \mathcal{U}_\alpha \xrightarrow{\sim} \mathring{D}^{i_\alpha}_{j_\alpha}(\epsilon) \times \mathbb{H}^{m-i_\alpha}$$

for some $i_\alpha \leq m-1$ and $1 \leq j_\alpha \leq n_i$, where $\mathbb{H}^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : x_k \geq 0\}$ is the closed half-plane.

8 Definition. If M and M' are $\widetilde{X}$-manifolds in $\mathbb{R}^m$, an $\widetilde{X}$*-cobordism* between them is an $\widetilde{X}$-manifold with boundary $K \subset \mathbb{R}^m \times I \subset \mathbb{R}^{m+1}$ such that $\partial K \subset \mathbb{R}^m \times \{0,1\}$, and for some $\delta > 0$ we have:

$$K \cap (\mathbb{R}^m \times [0,\delta)) = M \times [0,\delta)$$
$$K \cap (\mathbb{R}^m \times (1-\delta,1]) = M' \times (1-\delta,1]$$

as sets and all charts of the form

$$\mathcal{U} \times [0,\delta) \xrightarrow{\phi\times\mathrm{id}} \mathring{D}^i_j(\epsilon_1) \times \mathbb{R}^{m-i} \times [0,\delta) \xrightarrow{\sim} \mathring{D}^i_j(\epsilon_1) \times \mathbb{H}^{m-i+1}$$
$$\mathcal{V} \times (1-\delta,1] \xrightarrow{\psi\times\mathrm{id}} \mathring{D}^k_l(\epsilon_2) \times \mathbb{R}^{m-k} \times (1-\delta,1] \xrightarrow{\sim} \mathring{D}^k_l(\epsilon_1) \times \mathbb{H}^{m-k+1}$$

are boundary charts compatible with the atlas of K, where ϕ, ψ are charts of M and M' respectively and the last maps are given by the diffeomorphism $\mathbb{R}^n \times [0,1) \cong \mathbb{H}^{n+1}$.

2.4 $\widetilde{X}$-morphisms

The following notion of morphism is almost certainly too rigid for general use, however it is appropriate for the purposes of this paper. Furthermore, we will only use it in the case of isomorphism:

9 Definition. Given $\widetilde{X}$-manifolds $M \subset \mathbb{R}^m$ and $N \subset \mathbb{R}^n$, an $\widetilde{X}$-morphism $f : M \to N$ is a smooth map $f : M \to N$ such that for every chart $(\mathcal{V}, \psi)$ in the atlas of N. $(f^{-1}(\mathcal{V}), \psi \circ f|_{f^{-1}(\mathcal{V})})$ is a chart compatible with the atlas of M.

Applying compactness gives us the following lemma:

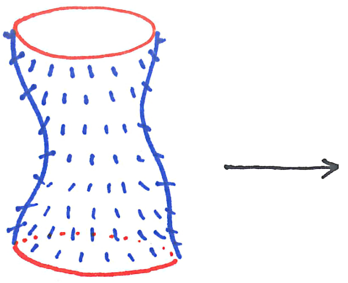


Figure 5: A $\widetilde{D^2}$-morphism.

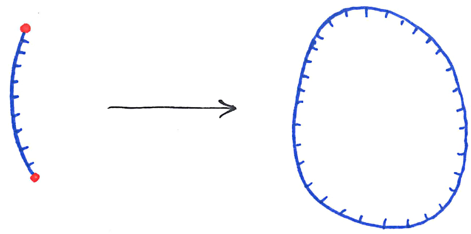


Figure 6: Not a $\widetilde{D^2}$-morphism.

10 Lemma. If M is an $\widetilde{X}$-manifold and M' is an $\widetilde{X}$-manifold with the same underlying set endowed with a sub-atlas of M (an $\widetilde{X}$-atlas that is a subcollection of the atlas of M), then $M \cong M'$. In particular, M is isomorphic to an $\widetilde{X}$-manifold with a finite atlas, by compactness.

11 Proposition. $\widetilde{X}$-isomorphic manifolds are $\widetilde{X}$-cobordant.

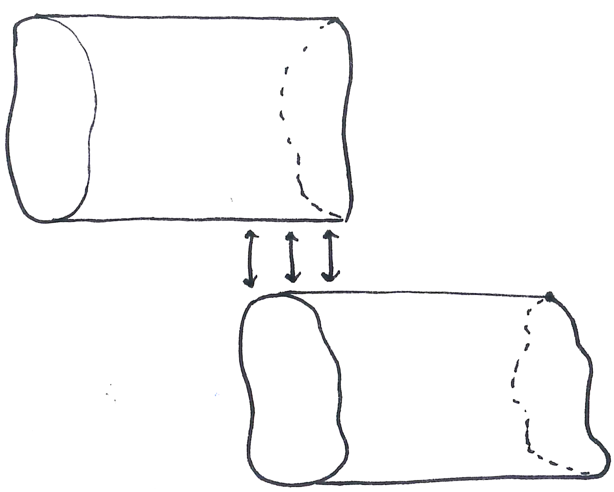
Proof. Let $\phi : M \to M'$ be an $\widetilde{X}$-isomorphism between $\widetilde{X}$-manifolds in $\mathbb{R}^m$. We build the cobordism by a kind of mapping cylinder construction: let K be the quotient of the space $M \times [0,1] \times \{0,1\}$ by the relation defined by $(x,t,0) \sim (x',t',1)$ if $f(x) = x'$ and $t = t' + \frac{1}{2}$. The quotient is homeomorphic to the mapping cylinder of f, and embeds naturally in $\mathbb{R}^m \times [0,1]$.

We endow K with a $\widetilde{X}$-atlas: let the charts consist of the following:

$$\mathcal{U} \times [0,1) \times \{0\} \to \mathcal{U} \times [0,1) \xrightarrow{\phi \times \mathrm{id}} \mathring{D}^i_j(\epsilon) \times \mathbb{R}^{m-i} \times [0,1) \xrightarrow{\sim} \mathring{D}^i_j(\epsilon) \times \mathbb{H}^{m-i+1}$$

$$\mathcal{V} \times [0,1) \times \{1\} \to \mathcal{V} \times [0,1) \xrightarrow{\psi \times \mathrm{id}} \mathring{D}^k_l(\delta) \times \mathbb{R}^{m-k} \times [0,1) \xrightarrow{\sim} \mathring{D}^k_l(\delta) \times \mathbb{H}^{m-k+1}$$

for charts ϕ of M and ψ of M' (the composites clearly descend to maps on the quotient space K). The compatibility of these charts then follows from the fact that f is an $\widetilde{X}$-isomorphism. $\square$



12 Proposition. $\widetilde{X}$-cobordism is an equivalence relation on the set of $\widetilde{X}$-manifolds in $\mathbb{R}^m$.

Proof. Symmetry is clear. Reflexivity can be checked by constructing a self-cobordism, for instance using charts of the form $\mathcal{U} \times [0, \frac{1}{2})$, $\mathcal{U} \times (\frac{1}{3}, \frac{2}{3})$, and $\mathcal{U} \times (\frac{1}{2}, 1]$. Transitivity follows from the fact that we can concatenate two cobordisms and glue pairs of boundary charts of the form $\mathcal{U} \times (1-\delta] \to \mathring{D}^i_j(\epsilon) \times \mathbb{H}^{m-i+1}$ and $\mathcal{U} \times [0, \delta) \to \mathring{D}^i_j(\epsilon) \times \mathbb{H}^{m-i+1}$ into non-boundary charts of the form $\mathcal{U} \times (1-\delta, 1+\delta) \to \mathring{D}^i_j(\epsilon) \times \mathbb{R}^{m-i+1}$. □

3 Transversality properties for $\widetilde{X}$-manifolds

We now study some key geometric properties of $\widetilde{X}$-manifolds. The first step is to find a family of subsets of X whose preimages under a continuous map of a manifold into X are submanifolds:

Let $f_{ij} := g_{ij}\big|_{\partial D^i}$ denote the attaching maps of the CW complex. Let $r_k : D^k \setminus \{*\} \to \partial D^k$ be the retraction. For each $\mathring{D}^i_j$, we iteratively construct a canonical open neighbourhood E^i_j, *the open star of* $\mathring{D}^i_j$, as follows:

$$
\begin{aligned}
E^i_j(0) :=& \mathring{D}^i_j \\
E^i_j(k+1) :=& E^i_j(k) \cup \bigcup_{l=1}^{n_k} g_{kl}\left(r_k^{-1}\left(f_{kl}^{-1}(E^i_j(k))\right)\right) \\
E^i_j :=& \bigcup_{k=i}^{\infty} E^i_j(k)
\end{aligned}
$$

Each $E^i_j(k)$ is open in X^k (by induction), thus E^i_j is an open neighbourhood of $\mathring{D}^i_j$. At each step we have retractions $E^i_j(k) \to E^i_j(k-1)$ consisting of the retractions on each cell. Chaining these together we get a retraction $r^i_j : E^i_j \to \mathring{D}^i_j$.

$f^{-1}(E^i_j)$ is an open subset of M, and so a manifold, whence the following definition makes sense:

13 Definition. Let V be a smooth manifold. A map $f : V \to X$ is *transverse* if for all i, j the map $r^i_j \circ f\big|_{f^{-1}(E^i_j)} : f^{-1}(E^i_j) \to \mathring{D}^i_j$ is smooth on a neighbourhood of $f^{-1}(c_{ij})$ and has c_{ij} as a regular value.

14 Definition. We say a CW complex X is *transversely constructed* if every attaching map $f_{ij} : \partial D^i \to X^{i-1}$ is transverse to X^{i-1}.

The following crucial result is an analogue of the Thom Transversality Theorem and is stated without proof:

15 Theorem. Suppose X is a transversely constructed CW complex. Then for any smooth manifold V and any map $f : V \to X$, there exists a map $f' : V \to X$ homotopic to f and transverse to X.

By applying this result skeleton by skeleton, starting with X^1, we can conclude:

16 Corollary. Every CW complex is homotopy equivalent to a transversely constructed one.

From now on, we will assume that X is transversely constructed. The *strata* of $\widetilde{X}$ are the sets $\widetilde{\Sigma}^i_j = (r^i_j)^{-1}(c_{ij}) \subset E^i_j \subset X$. Observe that $C^k_l = \bigcup_{i,j} \widetilde{\Sigma}^i_j \cap \mathring{D}^k_l$ and $\widetilde{X} = \bigcup_{i,j} \widetilde{\Sigma}^i_j$.

The strata themselves are not manifolds, however it turns out that their preimages are. The following Proposition echoes the standard transversality result on which the classical construction is based:

17 Proposition. If V is a smooth manifold and $f : V \to X$ a transverse map, $f^{-1}(\widetilde{\Sigma}^i_j)$ is a codimension i submanifold of V. Moreover, $f^{-1}(\widetilde{\Sigma}^i_j)$ has a canonical normal framing in V.

Proof. By the transversality condition on f, $f^{-1}(\widetilde{\Sigma}^i_j) = (r^i_j \circ f\big|_{f^{-1}(E^i_j)})^{-1}(c_{ij})$ is a codimension i submanifold of V, with the framing given by pulling back the canonical framing of $c_{ij} \in \mathring{D}^i_j \cong \mathbb{R}^i$. □

18 Theorem. If $f : (S^m, \infty) \to (X, x_0)$ is a based map transverse to X, viewing $S^m = \mathbb{R}^m \cup \{\infty\}$, $f^{-1}(\widetilde{X}) \subset \mathbb{R}^m$ is an $\widetilde{X}$-manifold.

Proof. Let $M := f^{-1}(\widetilde{X}) \subset \mathbb{R}^m$, it remains to find an $\widetilde{X}$-atlas for M that makes it into a $\widetilde{X}$-manifold. We first observe that M is compact: this follows from the fact that f is a pointed map and $\widetilde{X}$ is a closed subset of X not containing the base point x_0.

Since our CW complex is finite, we can start from the top dimensional cells $\mathring{D}^N_j$. Observe that $\widetilde{\Sigma}^N_j = \{c_{Nj}\}$, $E^N_j = \mathring{D}^N_j$, and $r^N_j = \mathrm{id}_{\mathring{D}^N_j}$ in this case. Thus, by the transversality condition on f, c_{Nj} is a regular value of $f\big|_{f^{-1}(\mathring{D}^N_j)}$. Therefore we have that every $x \in f^{-1}(c_{Nj})$ has an open neighbourhood $\mathcal{U}_j(x)$ on which f is a submersion onto some $\mathring{D}^N_j(\epsilon_j(x))$, moreover using the local form of a submersion we can choose the $\epsilon_j(x)$ small enough so that we have a diffeomorphism ϕ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{U}_j(x) \arrow[r, dashed, "\exists \phi \sim"] \arrow[dr, "f|_{\mathcal{U}_j(x)}"'] & \mathring{D}^N_j(\epsilon_j(x)) \times \mathbb{R}^{m-i} \arrow[d, "\pi"] \\
& \mathring{D}^N_j(\epsilon_j(x))
\end{tikzcd}$$

By slightly shrinking a chart if necessary, we can also ensure that it satisfies conditions 1-3. Thus we have obtained charts covering $f^{-1}(c_{Nj})$ for each j. Note that $\rho_N : X^N_{\text{punc}} \to X^{N-1}$ restricts to a map $\widetilde{X} \cap X^N_{\text{punc}} \to \widetilde{X}^{N-1}$. Then letting $f' := \rho_N \circ f : S^m \setminus f^{-1}(X^N_{\text{punc}}) \to X^{N-1}$, we have that $(f')^{-1}(\widetilde{X} \cap X^N_{\text{punc}}) = f^{-1}(\widetilde{X} \cap X^N_{\text{punc}})$ and f' is transverse to X^{N-1}, so we can apply the above construction again to obtain charts for the (preimages of) $(N-1)$-strata of $\widetilde{X}$, and so on. Once we have reached the bottom, the collection of charts that we have obtained will cover M.

It turns out that all charts so obtained that intersect are immediate to each other: indeed, for every chart $\phi : \mathcal{U} \to \mathring{D}^i_j \times \mathbb{R}^{m-i}$ that we have created, we have $\bar{\phi} = \rho^{N-i}_{i,N-i} \circ f\big|_{\mathcal{U}}$. The immediacy condition then follows due to the fact that $\rho^a_{i-k,k-n} \circ \rho^b_{i,k} = \rho^{a+b}_{i,k}$. □

4 The $\widetilde{X}$-cobordism group and the main result

As we have seen in Proposition 12, $\widetilde{X}$-cobordism induces an equivalence relation on the set of $\widetilde{X}$-manifolds in $\mathbb{R}^m$. We will call the resulting quotient $\Omega^{\widetilde{X}}_m$. The following fact is completely analogous to the classical case:

19 Proposition. $\Omega^{\widetilde{X}}_m$ is a group, under the operation of disjoint union in $\mathbb{R}^m$.

20 Theorem. $\pi_m(X) \cong \Omega^{\widetilde{X}}_m$.

We start by defining the map $d : [S^m, X] \to \Omega^{\widetilde{X}}_m$: for $[f] \in [S^m, X]$. Choose a representative f that is transverse to X (Theorem 15), apply Theorem 18 to obtain an $\widetilde{X}$-manifold $M \subset \mathbb{R}^m$, and define $d([f]) = M$. To check that this gives a well-defined map, it remains to check that given a pair of maps $f, f' : S^m \to X$ with a homotopy $H : S^m \times I \to X$, the $\widetilde{X}$-manifolds associated to f and f' are $\widetilde{X}$-cobordant. This follows by applying the same construction to the homotopy H but in "relative form": i.e. after replacing H by a transverse approximation, and deforming H to be constant on $S^m \times [0, \delta)$ and $S^m \times (1-\delta, 1]$ for some small δ, we construct an $\widetilde{X}$-atlas for $H^{-1}(\widetilde{X})$ as in Theorem 18 but this time subject to the constraint that we already have boundary charts defined on $\mathbb{R}^m \times [0, \delta)$ and $\mathbb{R}^m \times (1-\delta, 1]$. This ensures well-definedness of d up to cobordism, with respect to both homotopy of maps and the choices made in the construction.

The fact that d is a homomorphism is clear, by analogy with the classical case. It remains to prove that it is an isomorphism. We make need the notion of *manifold with corners*, see e.g. [5] for a reference. We will also make use of the following two properties:

21 Lemma (Straightening Corners)**.** Given a manifold with corners V, there is a manifold with boundary (and without corners) V' and a homeomorphism $h : V \xrightarrow{\sim} V'$ which restricts to a diffeomorphism away from the corner points of V.

22 Lemma. If V is a manifold with boundary and $\{W_i\}$ is a finite collection of codimension 0 submanifolds of V with boundary such that $\partial W_i \pitchfork \partial V$ for all i and $\partial W_i \pitchfork \partial W_j$ for all $i \neq j$, then $V \setminus \mathrm{Int}(\bigcup_i W_j)$ is a manifold with corners.

23 Proposition. d is surjective.

Proof. Let $M \subset \mathbb{R}^m$ be an $\widetilde{X}$-manifold. We construct a map $f : S^m \to X$ such that $d([f]) \simeq M$. By Lemma 10, we can assume that the atlas $\{(\mathcal{U}_\alpha, \phi_\alpha)\}$ of M is finite.

We start by defining f on the domain of the highest dimensional charts: let $\mathcal{F} = \overline{\bigcup_{i_\alpha = N} \mathcal{U}_\alpha} = \bigcup_{i_\alpha = N} \overline{\mathcal{U}_\alpha}$ (as there are only finitely many), then we can define f on $\mathcal{F}$ by letting $f\big|_{\mathcal{U}_\alpha} = \widetilde{\phi_\alpha}$ and extending to each $\overline{\mathcal{U}_\alpha}$. The immediacy condition guarantees that this is well-defined. Let $\mathcal{A} := \mathbb{R}^m \setminus \mathrm{Int}(\mathcal{F})$, by Lemma 22 this is a manifold with corners; our goal is now to extend f from $\partial\mathcal{A}$ to all of $\mathcal{A}$. We apply Lemma 21 to obtain a smooth structure without corners on $\mathcal{A}$, we then apply the Collar Neighbourhood Theorem to get a closed neighbourhood $\mathcal{C}$[1] of $\partial\mathcal{A}$ with $\mathcal{C} \cong \partial\mathcal{A} \times [0,1]$, moreover $\mathcal{C}$ can be chosen such that its boundary is transverse to the boundaries of all $(N-1)$-charts $\mathcal{V}$. Note that $f(\partial\mathcal{A}) \subseteq X^N_{\mathrm{punc}}$, so we may extend f to $\mathcal{C}$ by defining $f(x,t) = \rho_N^{(t)}(f(x))$. Letting $\mathcal{A}' = \mathcal{A} \setminus \mathrm{Int}(\mathcal{C})$, $\mathcal{A}'$ is a manifold with boundary and $f(\partial\mathcal{A}') \subseteq X^{N-1}$.

We can now apply induction on the dimension of X, since the above argument can be repeated with $\mathcal{A}'$ in place of $\mathbb{R}^m$ and $N-1$ in place of N, and so on. When we have reached the bottom, we will have f defined on some compact subset of $\mathbb{R}^m$ with f sending the boundary of this subset to x_0. Finally, we define f to be constantly x_0 on the rest of S^m.

Let $N := d(f)$, it remains to check that we have $N \simeq M$. First, we note that by the above construction we have $f^{-1}(\widetilde{X}) = M$, so the two are equal as sets. We show that the identity map $M \to N$ is an $\widetilde{X}$-isomorphism, possibly after rechoosing N, from which $M \simeq N$ follows by Proposition 11. To do this, we need the charts of M and N to be compatible: let $\psi : \mathcal{V} \to \mathring{D}^k_l(\epsilon_2) \times \mathbb{R}^{m-k}$ be a chart of N. By looking inside the construction of $d(f)$, we know that $\overline{\psi} = \rho^{N-k}_{k,N-k} \circ f\big|_{\mathcal{V}}$, where $\mathcal{V}$ is some neighbourhood of $f^{-1}(c_{kl})$. Moreover, we can redo the construction, this time choosing $\mathcal{V}$ small enough to be entirely contained in some k-chart ϕ of M. By well-definedness of d, N can be chosen to have this property for all charts $\mathcal{V}$. Then, by the construction of f, we have $f\big|_{\mathcal{V}} = \overline{\phi}$, so $\overline{\psi} = \rho^{N-k}_{k,N-k} \circ \overline{\phi}$ which is clearly compatible with all charts ϕ' of M, and vice versa. □

[1] We would also like to ensure that for any chart ψ defined on a region of $\mathcal{C}$, $\overline{\psi}(x,t) = \rho_N^{(t)}(f(x))$, which can be done by shrinking the N-charts slightly so that the collar lies entirely inside some N-chart, and requiring the parametrisation of the collar to agree with the radial parametrisation of that chart.

Injectivity of d follows by applying the same argument but in "relative form": given a cobordism K between $d(f)$ and $d(g)$, using the same techniques we can construct a map H that agrees with f and g on the boundary planes and which will serve as the required homotopy $f \simeq g$.

5 Examples & Applications

Before sketching some examples, it is helpful to review the relationship between $\widetilde{X}$-manifolds and normal framing.

24 Definition. For a stratum $\widetilde{\Sigma}^i_j$ of $\widetilde{X}$ and an $\widetilde{X}$-manifold $M \subset \mathbb{R}^m$, the $(m-i)$*-stratum of* M *corresponding to* $\widetilde{\Sigma}^i_j$ is the set

$$\Sigma^i_j = \bigcup_\alpha (\pi_\alpha \circ \phi_\alpha)^{-1}(\widetilde{\Sigma}^i_j)$$

where the union is taken over all charts, and π_α is the projection onto the first component. Each Σ^i_j is an $(m-i)$-submanifold of $\mathbb{R}^m$ with boundary $\bigcup_\beta \phi_\beta^{-1}\left((\widetilde{\Sigma}^i_j \cap \mathring{D}^{k_\beta}_{l_\beta}) \times \partial\mathbb{H}^{m-k_\beta}\right)$, where the union is taken over only the boundary charts.

Every $\pi \circ \phi : \mathcal{U} \to X$ is transverse to X, therefore every $(\pi \circ \phi)^{-1}(\widetilde{\Sigma}^i_j)$ is normally framed, by Proposition 17. Moreover, the chart compatibility conditions ensure that these framings glue together to give a framing of all of Σ^i_j. In places where the strata meet, their framings will be locally modelled on the behaviour of the strata in $\widetilde{X}$.

It is likely to be the case that a converse result can be formulated: that is to say, that an embedded stratified space with normal framings of its strata satisfying some compatibility conditions uniquely gives rise to an $\widetilde{X}$-manifold. Such a result can be viewed as an $\widetilde{X}$-analogue of the Tubular Neighbourhood Theorem. While we do not pursue this any further, it is very useful to informally adopt this "infinitesimal" perspective when working with examples, to avoid having to manually specify the $\widetilde{X}$-charts. We have already used this trick when presenting some of the examples in §2.

We will also use the trick of colouring each stratum of $\widetilde{X}$ or an $\widetilde{X}$-manifold with a unique colour. This graphically forbids the passage from one stratum to another in the absence of a third stratum that can mediate the transition.

5.1 Alternative CW decompositions of S^n

As we have already seen, taking X to be the cell structure on S^n consisting of one 0-cell and one n-cell, we recover the familiar theory. In a previous remark, we had noted that we will get a different construction if we choose a different cell structure, so we now investigate what happens when X is the cell structure of S^2 consisting of two 2-cells glued to a circle.

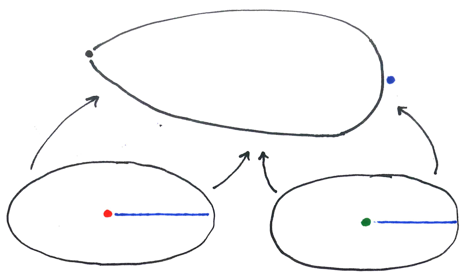


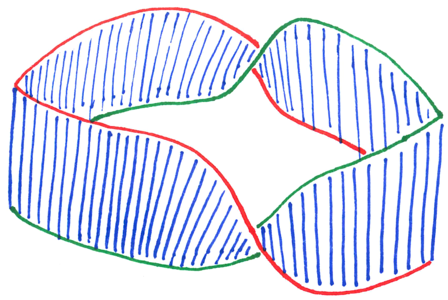
Figure 7: $\widetilde{X}$ when X consists of two 2-cells glued to a circle.

The generator of $\pi_2(S^2)$ looks like:

The generator of $\pi_3(S^2)$ looks like:

If we take an even more extravagant cell structure: with two 0-cells, two 1-cells, and two 2-cells, we get the following $\widetilde{X}$:

and the generators of π_2 and π_3 become, respectively:



5.2 $\mathbb{R}P^2$ and $\Sigma\mathbb{R}P^2$

This is the cobordism that trivialises twice the generator of $\pi_1(\mathbb{R}P^2)$:

$\Sigma\mathbb{R}P^2$ has a cell structure with a 3-cell attached to a sphere via the $\cdot 2$ map. The resulting $\widetilde{X}$ is:

Something interesting happens in $\pi_3(\Sigma\mathbb{R}P^2)$: first, observe that the Hopf map has order 2 in this group: this can also be seen from the cobordism picture by drawing the framed circle corresponding to the Hopf map $\mathfrak{h}$, introducing two red points, and then bringing them around the circle to annihilate each other on the opposite side. The resulting object is a circle with the framing twisted in the opposite direction, so we have shown that $\mathfrak{h} = -\mathfrak{h}$.

Next, we notice that $\mathfrak{h}$ can be factored as $2 \cdot \frac{\mathfrak{h}}{2}$, where $\frac{\mathfrak{h}}{2}$ is the following $\widetilde{X}$-manifold:

5.3 $S^n \vee S^m$

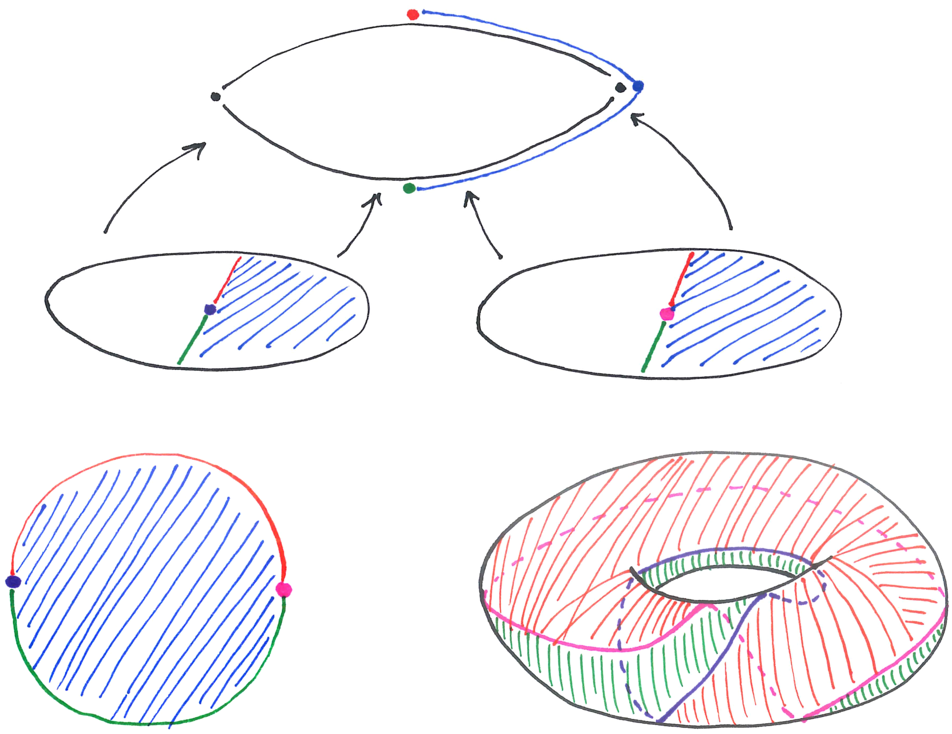
The last examples we consider are wedges of spheres. Taking the cell structure for $S^n \vee S^m$ consisting of one 0-cell, one n-cell, and one m-cell, $\widetilde{X}$ consists of two points: the centre points of the two spheres. Thus we see that $\pi_i(S^n \vee S^m)$ corresponds to pairs of disjoint framed submanifolds of $\mathbb{R}^i$ modulo pairs of disjoint framed cobordisms. The additional complexity of these groups comes from the fact that the manifolds can get tangled.

For example, $\pi_3(S^2 \vee S^2) \cong \mathbb{Z}^3$: the first two generators come from $\pi_3(S^2)$ via the two inclusions $S^2 \hookrightarrow S^2 \vee S^2$, while the third generator is:

i.e. the third $\mathbb{Z}$ factor corresponds to the linking number of the pair.

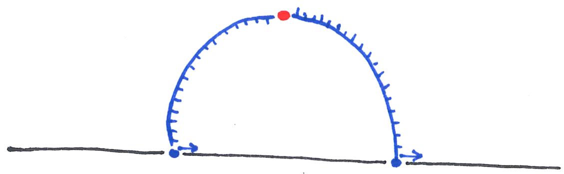
6 Conclusion

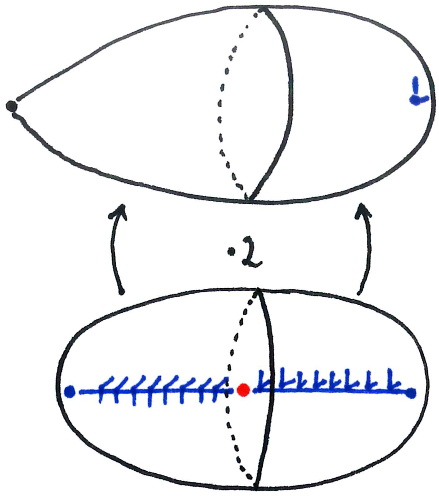
The result proved here heavily relies on the transverse approximation result of Theorem 15, whose proof could be the subject of future work. Another immediate avenue worth pursuing would be to complete the "infinitesimal" characterisation of $\widetilde{X}$-manifolds, which appears to be a natural way of viewing the objects studied here. To do this, we would need to prove some sort of converse to Definition 24: i.e. that an embedded stratified



space with normal framings of its strata, that also satisfies some compatibility conditions dictated by X, defines a unique $\widetilde{X}$-manifold.

An interesting project would be to apply the construction in this paper to the CW complex of a Thom space, and determine if the resulting $\widetilde{X}$-cobordism theory is manifestly equivalent to the cobordism theory of manifolds with the structure whose Thom space we started with.





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