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Chain Complexes as a Higher Category

1 Introduction

One of the 'philosophical' achievements of category theory is the creation of an effective language using which we can replace a claim of strict equality between objects with the exhibition of an isomorphism. If appropriate, this isomorphism could be shown to be unique or natural. Moreover, this style of reasoning yields vast practical benefits, especially in the fields of Algebra and Topology, where many arguments can be expressed in a conceptually efficient way using notions like naturality, universal property, adjunction, etc.

One thing that is missing from the definition of a (ordinary) category is the concept of *higher morphisms*, i.e. 'maps' (or 'relations') between ordinary morphisms. This bounds the extent to which we can enforce the 'categorical philosophy' described above. For instance, when asking that composition of morphisms be associative, we have to insist on strict equality of two morphisms (which after all, are often functions, and thus sets), in contravention of our program of doing away with strict equality. Therefore, it is natural to attempt to extend the definition of a category to one where we have in addition to objects and morphisms, 2-morphisms (which 'relate' morphisms). Continuing in this way, we would like to have a notion of n-morphisms (which 'relate' $(n-1)$-morphisms) for all n, and as a result, we hope to obtain something that we could call an ∞-category.

There are a two things to note at this stage. The first is that attempting a definition along the lines of the ordinary definition of a category (a collection of objects, for every pair of objects a set of 1-morphisms, for every pair of 1-morphisms a set of 2-morphisms, etc. such that the following conditions are satisfied...) quickly leads one to appreciate the unexpected complexity of the task at hand. To get a taste of this, observe what happens when one tries to relax associativity of composition of morphisms. Instead of $h \circ (g \circ f) = (h \circ g) \circ f$, we will instead have a family of natural 2-isomorphisms $a_{h,g,f} : h \circ (g \circ f) \xrightarrow{\sim} (h \circ g) \circ f$, which should satisfy a "pentagon identity" (familiar from the definition of weak monoidal categories). But commutativity of the pentagon is again an assertion of equality (this time among the $a_{h,g,f}$), so we need to replace it with yet another weakening in terms of 3-morphisms, and so on. Weakening associativity isn't the only problem, since in this setting it turns out that there is more than one way to compose n-morphisms: for instance 2-morphisms can be composed both 'vertically' and 'horizontally'. These different kinds of composition (called *pasting*) need to satisfy additional rules (but of course only up to higher isomorphism). This unsettling experience suggests to us that we should explore some other route to defining an ∞-category.

The second thing to note arises when considering examples of (hypothetical, for now) ∞-categories. One example is the category of topological spaces with morphisms the continuous maps, 2-morphisms the homotopies between maps, 3-morphisms the homotopies between homotopies, and so on. At this point we are reassured that our worrying about weak associativity in the preceding paragraph was not unfounded, as indeed the familiar composition of homotopies is well-defined only up to reparametrisation of homotopies, i.e. a 3-isomorphism. This example suggest another important feature: while all continuous maps need not be invertible, it happens to be the case that all homotopies (and homotopies between them, and so on) are invertible, i.e. all n-morphisms are isomorphisms for $n \geq 2$. Chain complexes (over an Abelian category), which are analogous in many ways to topological spaces, also appear to have this property. These examples lead us to look for a formalisation of ∞-categories with the property that all n-morphisms are invertible for $n \geq 2$. Such an object is often referred to as an $(\infty, 1)$-category.

There are multiple fruitful definitions of an $(\infty, 1)$-category, which include: complete Segal spaces, Segal categories, and quasi-categories. Substantial work has been done in relating these notions. Here we will examine the last of the three, which are also known as weak Kan complexes, and were first introduced by Boardman and Vogt in [1] and developed by Joyal in [2]. After explaining the definitions, we will work through the specific example of $Ch(\mathbf{RMod})$ as an ∞-category to verify that the desires laid out in the paragraphs above have been adequately satisfied.

Note that, throughout the report, we will ignore set-theoretical issues, such as size, in our definitions. If any foundational issues do arise from the assertions made here, the reader is assured that the definitions can be repaired by making suitable distinctions.

2 Definitions

2.1 Simplicial Sets

Definition 1. The *simplex category* $\mathit{\Delta}$ consists of the finite totally ordered sets, with order-preserving functions between them.

We will denote the totally ordered set $\{0 < 1 < \cdots < n\}$ by $[n]$.

Definition 2. A *simplicial object* in a category $\mathcal{A}$ is a contravariant functor $A : \mathit{\Delta} \to \mathcal{A}$.

Letting $A_n = A([n])$, and $A(\alpha) = \alpha^*$ for $\alpha : [m] \to [n]$, a simplicial object A can be viewed as a collection $\{A_n\}_{n\geq 0}$ of objects of $\mathcal{A}$ together with maps $\alpha^* : A_n \to A_m$ that are induced by the order-preserving functions $\alpha : [m] \to [n]$. We can simplify this presentation by observing the following:

Proposition 3. Let $\epsilon_i : [n] \to [n+1]$ be the *ith face map*, the order-preserving injective function with image $[n+1] - i$, and let $\eta_j : [n+1] \to [n]$ be the *j-th degeneracy map*, the order-preserving surjective function which maps two elements to i. Then every order-preserving function $\alpha : [m] \to [n]$ can be uniquely decomposed as $\alpha = \epsilon_{i_1}\epsilon_{i_2} \ldots \epsilon_{i_s}\eta_{j_1}\eta_{j_2} \ldots \eta_{j_t}$.

Another helpful result that will be stated without proof is:

Proposition 4. The face and degeneracy maps satisfy the following identities:

$$\epsilon_j\epsilon_i = \epsilon_i\epsilon_{j-1} \text{ if } i < j$$
$$\eta_j\eta_i = \eta_i\eta_{j+1} \text{ if } i \leq j$$
$$\eta_j\epsilon_i = \begin{cases} \epsilon_i\eta_{j-1} & \text{if } i < j \\ \text{id} & \text{if } i = j, j+1 \\ \epsilon_{i-1}\eta_j & \text{if } i < j+1 \end{cases}$$

Using these results, we can recharacterise a simplicial object as a collection $\{A_n\}_{n\geq 0}$ of objects of $\mathcal{A}$ together with *face operators* $\partial_i : A_n \to A_{n-1}$ and *degeneracy operators* $\sigma_i : A_n \to A_{n+1}$ (for $0 \leq i \leq n$), which satisfy the above identities. The equivalence of this characterisation follows from the existence and uniqueness of the decomposition and the functoriality of A.

The simplicial objects of $\mathcal{A}$ with natural transformations between them form a category $\mathbf{s}\mathcal{A}$. The most important case for our purposes are *simplicial sets* (where $\mathcal{A}$ is **Set**), which we will use to define ∞-categories.

As a 'building block' for **sSet**, we introduce the *standard n-simplex* $\Delta^n := \text{Hom}_{\mathit{\Delta}}(-, [n])$, which is a simplicial set, with $(\Delta^n)_i = \text{Hom}_{\mathit{\Delta}}([i], [n])$. This definition is motivated by the analogy between Δ^n and the *standard geometric n-simplex* $|\Delta^n| \subseteq \mathbb{R}^{n+1}$. If A is any simplicial set, using the Yoneda Lemma, we see that there is a natural bijection of sets

$$A_n \cong \text{Hom}_{\mathbf{sSet}}(\Delta^n, A)$$

which suggests the interpretation that the set A_n corresponds to the different ways of mapping Δ^n into A, i.e. the set of n-simplices in A. Accordingly, we will often refer to the elements of A_0 as vertices, elements of A_1 as edges, and so on.

Example 5. For a topological space X, there is the *singular simplicial set* $\operatorname{Sing} X$ where $(\operatorname{Sing} X)_n := \operatorname{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$ is the set of singular n-simplices in X. ∂_i associates to a singular n-simplex its ith face, and σ_i associates to an n-simplex the degenerate $(n+1)$-simplex with the vertex i repeated. This extends to a functor $\operatorname{Sing}(-) : \mathbf{Top} \to \mathbf{sSet}$. $\operatorname{Sing}(-)$ admits a left adjoint which takes a simplicial set A to a CW-complex $|A|$, its *geometric realisation.* The details of this construction are not needed for our purposes, but it is worth knowing that we can always picture a simplicial set geometrically (though most will be infinite dimensional).

2.2 Quasi-categories

Our aim is to define ∞-categories combinatorially, as a special kind of simplicial set. Before we do this, it is worth examining how a simplicial set can represent an ordinary category:

Definition 6. Let $\mathcal{C}$ be a category, then the *nerve of* $\mathcal{C}$ is the simplicial set $\mathrm{N}(\mathcal{C})$ defined by: $\mathrm{N}(\mathcal{C})_n = \operatorname{Hom}_{\mathbf{Cat}}([n], \mathcal{C})$ (where the poset $[n]$ is viewed as a category). Explicitly, the elements of $\mathrm{N}(\mathcal{C})_n$ are sequences $X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} X_n$ of n composable morphisms of $\mathcal{C}$, the ith face of which is the sequence:

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_{i-1}} X_{i-1} \xrightarrow{f_{i+1} \circ f_i} X_{i+1} \xrightarrow{f_{i+2}} \dots \xrightarrow{f_n} X_n$$

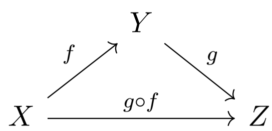
and the ith degeneracy is given by:

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \dots \xrightarrow{f_i} X_i \xrightarrow{\mathrm{id}_{X_i}} X_i \xrightarrow{f_{i+1}} \dots \xrightarrow{f_n} X_n$$

The nerve construction is our first taste of how categorical structure can be encoded using simplicial sets. In fact, we can completely recover the structure of $\mathcal{C}$ from $\mathrm{N}(\mathcal{C})$: the objects of $\mathcal{C}$ are the vertices of the nerve and the morphisms the edges. To recover composition, we observe that two appropriately oriented edges $\phi = X \xrightarrow{f} Y$ and $\psi = Y \xrightarrow{g} Z$ in the nerve will share a common face $\tau = X \xrightarrow{f} Y \xrightarrow{g} Z$, and the desired composite is the edge $\partial_1(\tau) = X \xrightarrow{g \circ f} Z$. This simplicial interpretation of morphism composition motivates the following definition:

Definition 7. The *kth horn* Λ^n_k of Δ^n is the simplicial set obtained by removing from Δ^n the single (non-degenerate) n-cell and its jth face (along with all of their degeneracies). Formally, $(\Lambda^n_k)_i \subseteq (\Delta^n)_i = \operatorname{Hom}_{\Delta}([i], [n])$ consists of the order-preserving functions $f : [i] \to [n]$ such that $\{k\} \cup f([i]) \neq [n]$.

We have just seen (the image of) the horn Λ^2_1 appear in the nerve construction above, as the composable pair of edges ϕ and ψ.



Crucially, we relied on the fact that ϕ and ψ were the edges of a face τ, whose third edge represented the composite $g \circ f$. Formally, this means that the map of the horn $\Lambda^2_1 \to \mathrm{N}(\mathcal{C})$ can be extended to a map of the whole triangle $\Delta^2 \to \mathrm{N}(\mathcal{C})$, i.e. that we can 'fill' the horn. This leads us to the thought that in order to have a notion of composition, this must always be possible, which is the content of the main definition of this paper:

Definition 8. A *quasi-category* (or *weak Kan complex*) is a simplicial set K with the additional property that every map $\Lambda^n_k \to K$ with $0 < k < n$ can be extended to a map $\Delta^n \to K$.

There two technical aspects of this definition that are worth commenting on. Firstly, the 'outer horns' Λ^n_0 and Λ^n_n are not required to be 'fillable', meaning that they don't correspond to morphisms that need to be composable. In the nerve example above where $n = 2$, this reflects the fact that we should be able to compose g and f but given some $f : X \to Y$ and $h : X \to Z$ we don't always need to be able to factor $h = g \circ f$ for some g. A simplicial set in which *all* horns can be filled is called a *Kan complex.*

Secondly, the extension need not be unique, unlike the nerve example. As we shall see shortly, in a quasi-category, given morphisms (edges) f and g there could be more than one edge representing their composite. In fact, if it is the case that all of the fillers are unique, then K must be the nerve of some ordinary category[1].

3 Low-Dimensional Behaviour

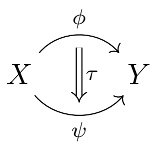
In order to better understand this definition, we will now work through the low-dimensional behaviour. The objects of our quasi-category are the vertices of K and the edges correspond to 1-morphisms: if $\phi \in K_1$ then ϕ can be thought of as a morphism $\partial_0(\phi) \to \partial_1(\phi)$.

Definition 9. Let K be a quasi-category, and let $X, Y \in K_0$.

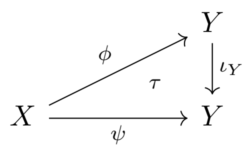
(i) $\phi \in K_1$ is a *morphism* $X \to Y$ if $\partial_0(\phi) = X$ and $\partial_0(\phi) = Y$.
(ii) the *identity morphism at* X is the degenerate 1-simplex $\iota_X := \sigma_0(X)$.
(iii) for $Z \in K_0$ and morphisms $\phi : X \to Y$ and $\psi : Y \to Z$, we write $\psi \circ \phi$ for any morphism $\chi : X \to Z$ such that there exists $\tau \in K_2$ with $\partial_0(\tau) = \psi$, $\partial_1(\tau) = \chi$, and $\partial_2(\tau) = \phi$.

Remark. The existence of at least one such morphism $\chi : X \to Z$ is guaranteed by the inner horn condition.

We turn to 2-morphisms, which are exhibited by 2-cells: if ϕ and ψ are edges between the same two vertices, i.e. $\partial_0(\phi) = \partial_0(\psi) = X$ and $\partial_1(\phi) = \partial_1(\psi) = Y$, then a 2-morphism $\phi \Rightarrow \psi$ is a 'collapsed' 2-simplex $\tau \in K_2$ such that $\partial_0(\tau) = \iota_Y$, $\partial_1(\tau) = \psi$, and $\partial_2(\tau) = \phi$.



is represented by:



Definition 10. Let K be a quasi-category. For $\phi, \psi : X \to Y$ morphisms of K, we say $\tau \in K_2$ *is a* 2*-morphism (or homotopy)* $\phi \Rightarrow \psi$ if $\partial_0(\tau) = \iota_Y$, $\partial_1(\tau) = \psi$, and $\partial_2(\tau) = \phi$, and we write $\phi \simeq \psi$. The *identity* 2*-morphism at* ϕ is the degenerate 2-simplex $j_\phi = \sigma_1(\phi)$.

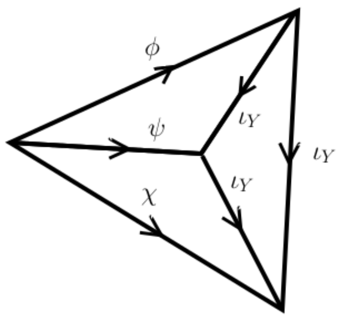
Proposition 11. For objects $X, Y \in K_0$ and morphisms $\phi, \psi : X \to Y$, we have:

(i) $\phi \simeq \phi$
(ii) $\iota_Y \circ \phi \simeq \phi$
(iii) $\phi \circ \iota_X \simeq \phi$
(iv) if $\phi \simeq \psi$ and $\psi \simeq \chi$, then $\phi \simeq \chi$.
(v) if $\phi \simeq \psi$ then $\psi \simeq \phi$.

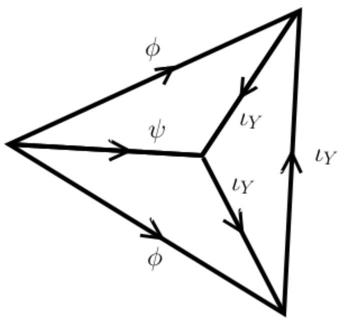
Proof. (i), (ii), and (iii) follow immediately from the definitions.

[1] For a proof of this, see [5].

(iv) $\iota_Y \simeq \iota_Y$, so we can apply the inner horn condition to the following horn:

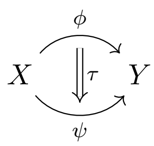


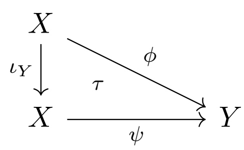
(v) Apply the inner horn condition to the following horn, and then apply (iv).



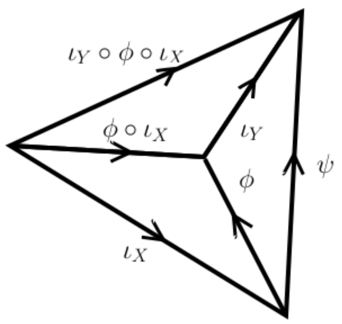
□

So $\simeq$ is an equivalence relation, and in particular is symmetric, so every 2-morphism is a 2-isomorphism. We could have alternatively defined $\simeq$ so that:



is represented by:


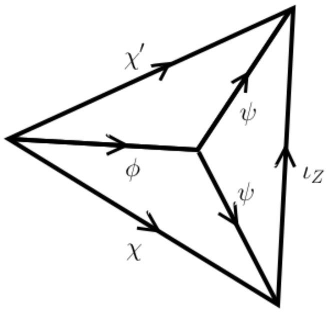
It is not hard to see that we would have obtained the same relation: we deduce this by applying the inner horn condition and (ii), (iii), and (iv) above to the following horn:



We can now show that composition, while not uniquely defined, is well-defined up to 2-isomorphism:

Proposition 12. If χ and χ' are two composites $\psi \circ \phi$, then $\chi \simeq \chi'$.

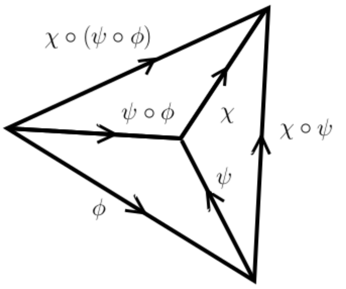
Proof. Apply the inner horn condition to: □



Weak associativity now follows easily:

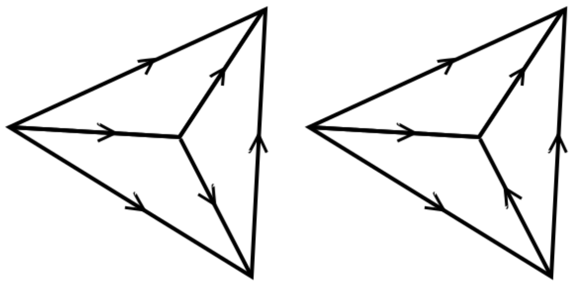
Proposition 13. For objects $X, Y, Z, W \in K_0$ and morphisms $\phi : X \to Y$, $\psi : Y \to Z$, $\chi : Z \to W$, $\chi \circ (\psi \circ \phi) \simeq (\chi \circ \psi) \circ \phi$.

Proof. Apply the inner horn condition to:

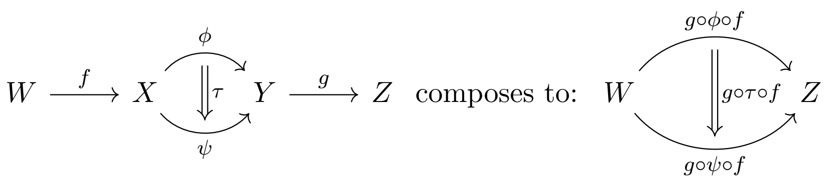


Thus $\chi \circ (\psi \circ \phi)$ is a composite $(\chi \circ \psi) \circ \phi$, so by the previous Proposition, $\chi \circ (\psi \circ \phi) \simeq (\chi \circ \psi) \circ \phi$. □

Now that we have understood composition of morphisms and the relation $\simeq$, what can be said about composition of 2-morphisms? By applying the inner horn condition to the horns Λ^3_1 and Λ^3_2, we get two distinct arrangements of three 2-morphisms that can be composed into a single 2-morphism. We have already used both of these compositions to prove some of the Propositions about $\simeq$ above. The first of these gives vertical composition as a special case (when one of the 2-



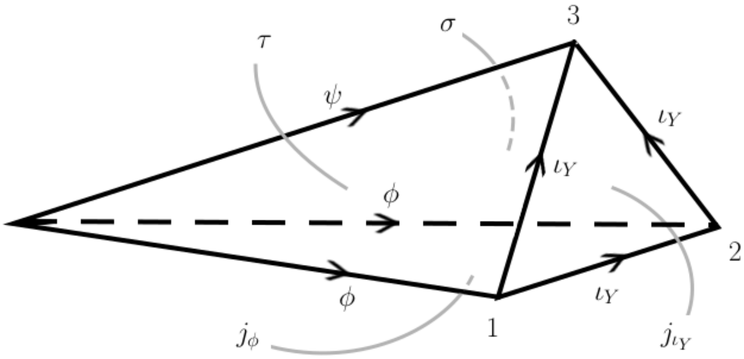
morphisms is taken to be the identity), while the second gives horizontal composition. By letting two of the 2-morphisms be the identity, we can also define the composition of a 2-morphism with a 1-morphism (in two different ways), which is called *whiskering*:



Motivated by the same geometric intuition, we can make the following definition:

Definition 14. Given $X, Y \in K_0$, morphisms $\phi, \psi : X \to Y$, and two 2-morphisms $\tau, \upsilon : \phi \Rightarrow \psi$, we say that $\rho \in K_3$ is a 3-*morphism* $\tau \Rrightarrow \upsilon$ if:

$$\begin{aligned} \partial_0(\rho) &= j_{\iota_Y} \\ \partial_1(\rho) &= \upsilon \\ \partial_2(\rho) &= \tau \\ \partial_3(\rho) &= j_\phi \end{aligned}$$



As before, we can prove that the relation that this defines is an equivalence relation, this time by appealing to the existence of 4-dimensional horn fillers. Using this, we can prove that the (horizontal/vertical) composition of 2-morphisms is weakly associative and well-defined up to 3-isomorphism. We can also prove the "interchange rule" using similar arguments.

Of course, the purpose of this definition of ∞-category is to guarantee all of these properties, as well as their generalisations to all higher dimensions. By working through the low-dimensional examples, we have seen how this simple combinatorial definition subsumes the myriad of complicated conditions mentioned in the introduction.

4 Constructing ∞-Categories

We have examined the behaviour of morphisms in an abstract quasi-category, which agree with our intuition of what an ∞-category should look like. However, it is not immediately clear how we can endow an ordinary category with the necessary simplicial set structure to make it the ∞-category we want. As we saw, the nerve $\mathrm{N}(\mathcal{C})$ of an ordinary category $\mathcal{C}$ is a quasi-category, but does not carry the additional structure we are looking for: all n-morphisms (as we defined them in the previous section) are trivial for $n \geq 2$. To obtain a non-trivial quasi-category, we will need to provide more structure.

We will show how to obtain quasi-categories in two different ways: from simplicially enriched categories and differential graded categories. The former is a more flexible approach, which we will now outline but not pursue. The latter will be more convenient to apply to the example of chain complexes, which we study in the next section.

Definition 15. A *simplicial category* is a category enriched over **sSet**. $\mathbf{Cat}_\Delta$ will denote the category of (small) simplicial categories.

As a starting point, by analogy with how we used $[n]$ in our definition of the nerve, we define a simplicial category $\mathbb{C}[\Delta^n]$.

Definition 16. The simplicial category $\mathbb{C}[\Delta^n]$ is defined as follows: the objects of $\mathbb{C}[\Delta^n]$ are the objects of $[n]$. For $i, j \in [n]$, have:

$$\mathrm{Hom}_{\mathbb{C}[\Delta^n]}(i,j) = \begin{cases} \emptyset & \text{if } i > j \\ \mathrm{N}(P_{i,j}) & \text{if } i \leq j \end{cases}$$

where the poset $P_{i,j} := \{I \subset \{i, i+1, \ldots, j\} : i, j \in I\}$ is viewed as a category and N is the nerve defined earlier. Composition is induced by the map $(P_{i,j}, P_{j,k}) \mapsto P_{i,j} \cup P_{j,k}$.

$\mathbb{C}[\Delta^n]$ is just $[n]$ with more morphisms: for $i \leq j$, instead of exactly one morphism $i \to j$, $\mathbb{C}[\Delta^n]$ has one morphism for each increasing sequence $i = i_0 < \cdots < i_k = j$ (and the collection of these increasing sequences is a simplicial set).

Definition 17. The *simplicial nerve* of a simplicial category $\mathcal{C}$ is the simplicial set $\mathrm{N}_s(\mathcal{C})$ given by $\mathrm{N}_s(\mathcal{C})_n = \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, \mathrm{N}_s(\mathcal{C})) := \mathrm{Hom}_{\mathbf{Cat}_\Delta}(\mathbb{C}[\Delta^n], \mathcal{C})$.

Theorem 18. Let $\mathcal{C}$ be a simplicial category such that every Hom-set is a Kan complex. Then $\mathrm{N}_s(\mathcal{C})$ is a quasi-category.

Proof. See [5]. □

5 Chain Complexes

We are now ready to turn the category of chain complexes of R-modules into a quasi-category, by showing that they form a differential graded category. Another more general method uses the Dold-Kan correspondence (see [3]) to associate to each pair of chain complexes a simplicial set, which turns out to be a Kan complex, and then applying the simplicial nerve construction from Section 4 to obtain a quasi-category. Here we will use a different method which lends itself particularly well with dealing with chain complexes and which will make it easier to explicitly work out the low dimensional morphisms[2].

5.1 Preliminaries

Definition 19. A *differential graded category* $\mathcal{C}$ consists of a collection of objects, for every two objects X and Y a chain complex of Abelian groups $\mathrm{Hom}_{\mathcal{C}}(X, Y)_\bullet$, a collection of bilinear maps

$$\circ : \mathrm{Hom}_{\mathcal{C}}(Y, Z)_q \times \mathrm{Hom}_{\mathcal{C}}(X, Y)_p \to \mathrm{Hom}_{\mathcal{C}}(X, Z)_{p+q}$$

that satisfy the "Leibniz rule" $d(g \circ f) = dg \circ f + (-1)^q g \circ df$ and are associative so that $(h \circ g) \circ f = h \circ (g \circ f)$, and for each object X, identity maps $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)_0$ such that $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$ for all $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)_p$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)_q$.

Similarly to simplicial categories, there is a nerve construction that allows us to obtain a quasi-category from any differential graded category:

Definition 20. Let $\mathcal{A}$ be a differential graded category, then the *differential graded nerve* is the simplicial set $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ defined as follows: $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_0$ consists of the objects of $\mathcal{C}$, and for $n \geq 1$ $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n$ is the set of all ordered pairs $(\{X_i\}_{0 \leq i \leq n}, \{f_I\}_I)$ where each X_i is an object of $\mathcal{C}$ and for a subset $I = \{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\} \subseteq [n]$, f_I is an element of $\mathrm{Hom}_{\mathcal{C}}(X_{i_-}, X_{i_+})_m$ that satisfies:

$$df_I = \sum_{1 \leq j \leq m} (-1)^j (f_{I - \{i_j\}} - f_{\{i_j < \cdots < i_1 < i_+\}} \circ f_{\{i_- < i_m \cdots < i_j\}})$$

where the sum is 0 when $m = 0$.

The ith face map $\partial_i : \mathrm{N}_{\mathrm{dg}}(\mathcal{C})_n \to \mathrm{N}_{\mathrm{dg}}(\mathcal{C})_{n-1}$ is given by:

$$\partial_i \Big((\{X_i\}_{0 \leq i \leq n}, \{f_I\}_I) \Big) = (\{X_{\alpha(j)}\}_{0 \leq j \leq (n-1)}, \{f_{\epsilon_i(J)}\}_J)$$

[2]See Proposition 1.3.1.17 of [7] for a proof that the two constructions yield equivalent results, in a sense that is defined in [5].

This combinatorial definition looks obscure, but is fairly easy to compute for small n in concrete examples, as we shall see. The following result, found in [7][3], assures us that we get a quasi-category from any differential graded category. Moreover, the construction can be used to obtain explicit expressions for the composition of morphisms in a differential graded category.

Theorem 21. The simplicial set $\mathrm{N}_{\mathrm{dg}}(\mathcal{C})$ is a quasi-category for any differential graded category $\mathcal{C}$.

Proof. Let $K = \mathrm{N}_{\mathrm{dg}}(\mathcal{C})$. We need to show that every map $\Lambda^n_k \to K$ extends to a map $\Delta^n \to K$ when $0 < k < n$. The kth horn corresponds to a pair $(\{X_i\}_{0 \le i \le n}, \{f_I\}_I)$ where I ranges over subsets $\{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\} \subseteq [n]$ such that $I \neq [n], [n] - \{k\}$ and where all of the $f_I \in \mathrm{Hom}_{\mathcal{C}}(X_{i_-}, X_{i_+})_m$ satisfy:

$$df_I = \sum_{1 \le j \le m} (-1)^j (f_{I - \{i_j\}} - f_{\{i_j < \cdots < i_1 < i_+\}} \circ f_{\{i_- < i_m \cdots < i_j\}}) \qquad (\star)$$

This can be extended to a pair $(\{X_i\}_{0 \le i \le n}, \{f_I\}_I)$ where I now ranges over all subsets $\{i_- < i_m < i_{m-1} < \cdots < i_1 < i_+\} \subseteq [n]$ by setting:

$$f_{[n]} = 0 \qquad (1)$$

$$f_{[n]-\{k\}} = \sum_{0<j<n} (-1)^{j-k} f_{\{j, j+1, \ldots, n\}} \circ f_{\{0, \ldots, j\}} - \sum_{\substack{0<j<n \\ j \neq k}} (-1)^{j-k} f_{[n]-\{j\}} \qquad (2)$$

Then $df_{[n]} = 0$, which satisfies $(\star)$ and we can check using the Leibniz rule that $df_{[n]-\{k\}}$ satisfies $(\star)$ as well, so $(\{X_i\}_{0 \le i \le n}, \{f_I\}_I)$ is the desired n-simplex. ☐

We will now show that $Ch(\mathbf{RMod})$ forms a differential graded category, so that $\mathrm{N}_{\mathrm{dg}}(Ch(\mathbf{RMod}))$ forms a quasi-category.

Proposition 22. The objects of $Ch(\mathbf{RMod})$ form a differential graded category, where to any two chain complexes $A_\bullet, B_\bullet$ we assign the *mapping complex* $\underline{\mathrm{Hom}}(A_\bullet, B_\bullet)_*$, a chain complex of Abelian groups where:

$$\underline{\mathrm{Hom}}(A_\bullet, B_\bullet)_n := \prod_{m \in \mathbb{Z}} \mathrm{Hom}(A_m, B_{n+m})$$

and the differentials are given by: $(d_n(f))_m := d^B_{n+m} \circ f_m - (-1)^n f_{m-1} \circ d^A_m$ where $f = (f_m) \in \prod_{m \in \mathbb{Z}} \mathrm{Hom}(A_m, B_{n+m})$. The bilinear composition $\circ : \underline{\mathrm{Hom}}(B_\bullet, C_\bullet)_q \times \underline{\mathrm{Hom}}(A_\bullet, B_\bullet)_p \to \underline{\mathrm{Hom}}(A_\bullet, C_\bullet)_{p+q}$ is given by:

$$(g \circ f)_m := g_{p+m} \circ f_m$$

which clearly satisfies $(h \circ (g \circ f))_m = ((h \circ g) \circ f)_m$, and we can check the Leibniz rule:

$$\begin{aligned}
d_{p+q}(g \circ f)_m &= d^C_{p+q+m} \circ (g \circ f)_m - (-1)^{p+q} (g \circ f)_{m-1} \circ d^A_m \\
&= d^C_{p+q+m} \circ g_{p+m} \circ f_m - (-1)^{p+q} g_{p+m-1} \circ f_{m-1} \circ d^A_m \\
&= \left(d^C_{p+q+m} \circ g_{p+m} - (-1)^q g_{p+m-1} \circ d^B_{p+m}\right) \circ f_m \\
&\quad + (-1)^q g_{p+m-1} \circ \left(d^B_{p+m} \circ f_m - (-1)^p \circ f_{m-1} \circ d^A_m\right) \\
&= (dg)_{p+m} \circ f_m + (-1)^q g_{p+m-1} \circ (df)_m \\
&= (dg \circ f)_m + (-1)^q (g \circ df)_m
\end{aligned}$$

Remark. The mapping complex $\underline{\mathrm{Hom}}(A_\bullet, B_\bullet)$ used above is just the total complex $\mathrm{tot}_\Pi(\mathrm{Hom}(A_\bullet, B_\bullet))$.

[3] At the time of writing, the construction given in [7] contained an error, which is fixed here.

First, we describe $\mathrm{N_{dg}}(Ch(\mathbf{RMod}))_n$ for $n \leq 2$. $\mathrm{N_{dg}}(Ch(\mathbf{RMod}))_0$ consists of the objects of $Ch(\mathbf{RMod})$. $\mathrm{N_{dg}}(Ch(\mathbf{RMod}))_1$ consists of the pairs $(\{A_\bullet, B_\bullet\}, \{f\})$ where $f = (f_m) \in \underline{\mathrm{Hom}}(A_\bullet, B_\bullet)_0$ satisfies $df = 0$, i.e. $d^B_m \circ f_m = f_{m-1} \circ d^A_m$ for all m, and we have:

$$\partial_0 \left(\left(\{A_\bullet, B_\bullet\}, \{f\} \right) \right) = A_\bullet$$
$$\partial_1 \left(\left(\{A_\bullet, B_\bullet\}, \{f\} \right) \right) = B_\bullet$$

In other words, $\mathrm{N_{dg}}(Ch(\mathbf{RMod}))_1$ consists of the chain maps $f : A_\bullet \to B_\bullet$ and the face operators assign to f its domain and codomain.

$\mathrm{N_{dg}}(Ch(\mathbf{RMod}))_2$ consists of pairs $(\{A_\bullet, B_\bullet, C_\bullet\}, \{f, g, h, z\})$ where $f : A_\bullet \to B_\bullet, g : B_\bullet \to C_\bullet, h : A_\bullet \to C_\bullet$ are chain maps and $z : A_\bullet \to C_{\bullet+1}$ satisfies $dz = g \circ f - h$, i.e. $d^C_{m+1} \circ z_m + z_{m-1} \circ d^A_m = g_m \circ f_m - h_m$. The face operators are:

$$\partial_0 \left(\left(\{A_\bullet, B_\bullet, C_\bullet\}, \{f, g, h, z\} \right) \right) = \left(\{A_\bullet, B_\bullet\}, \{f\} \right)$$
$$\partial_1 \left(\left(\{A_\bullet, B_\bullet, C_\bullet\}, \{f, g, h, z\} \right) \right) = \left(\{A_\bullet, C_\bullet\}, \{h\} \right)$$
$$\partial_2 \left(\left(\{A_\bullet, B_\bullet, C_\bullet\}, \{f, g, h, z\} \right) \right) = \left(\{B_\bullet, C_\bullet\}, \{g\} \right)$$

5.2 n-morphisms in $Ch(\mathbf{RMod})$

We are now ready to describe $\mathrm{N_{dg}}(Ch(\mathbf{RMod}))$ as a quasi-category with the language developed in Section 3. From the above description, a morphism $\phi : A_\bullet \to B_\bullet$ consists of a chain map $\phi_f : A_\bullet \to B_\bullet$. The identity morphism $\iota_{A_\bullet}$ is the identity chain map.

Given morphisms $\phi, \psi : A_\bullet \to B_\bullet$, a 2-morphism $\tau : \phi \Rightarrow \psi$ is a pair $(\{A_\bullet, B_\bullet\}, \{\phi_f, \psi_f, \mathrm{id}_{B_\bullet}, \tau_z\})$ where $\tau_z : A_\bullet \to B_{\bullet+1}$ satisfies:

$$d_{m+1} \circ (\tau_z)_m + (\tau_z)_{m-1} \circ d_m = (\phi_f)_m - (\psi_f)_m$$

i.e. τ_z a chain homotopy from ϕ_f to ψ_f. The identity 2-morphism at ϕ is

$$j_\phi = \left(\{A_\bullet, B_\bullet\}, \{\phi_f, \phi_f, \mathrm{id}_{B_\bullet}, 0\} \right)$$

We can now construct something new: recall that in Definition 14 we introduced the concept of a 3-morphism $\rho : \tau \Rrightarrow \upsilon$. We can find out what this means for chain complexes, but first we must compute $\mathrm{N_{dg}}(Ch(\mathbf{RMod}))_3$: after some unpacking, one finds that it consists of pairs $\rho = (\{A_\bullet, B_\bullet, C_\bullet, D_\bullet\}, \{f, g, h, i, j, k, x, y, z, w, \Omega\})$ where:

- $f : A_\bullet \to B_\bullet$, $g : A_\bullet \to C_\bullet$, $h : A_\bullet \to D_\bullet$, $i : B_\bullet \to C_\bullet$, $j : B_\bullet \to D_\bullet$, $k : C_\bullet \to D_\bullet$ are chain maps;
- $x : A_\bullet \to C_{\bullet+1}$, $y : A_\bullet \to D_{\bullet+1}$, $z : A_\bullet \to D_{\bullet+1}$, $w : B_\bullet \to D_{\bullet+1}$ satisfy:

$$dx = i \circ f - g$$
$$dy = j \circ f - h$$
$$dz = k \circ g - h$$
$$dw = k \circ i - j$$

- $\Omega : A_\bullet \to D_{\bullet+2}$ satisfies $d\Omega = k \circ x - y + z - w \circ f$.

The face maps are

$$\partial_0(\rho) = (\{A_\bullet, B_\bullet, C_\bullet\}, \{i, j, k, w\}))$$
$$\partial_1(\rho) = (\{A_\bullet, C_\bullet, D_\bullet\}, \{g, h, k, z\}))$$
$$\partial_2(\rho) = (\{A_\bullet, B_\bullet, D_\bullet\}, \{f, h, j, y\}))$$
$$\partial_3(\rho) = (\{A_\bullet, B_\bullet, C_\bullet\}, \{f, g, i, x\}))$$

So for 2-morphisms $\tau = ((\{A_\bullet, B_\bullet\}, \{\phi_f, \psi_f, \mathrm{id}_{B_\bullet}, \tau_z\}))$ and $\upsilon = ((\{A_\bullet, B_\bullet\}, \{\phi_f, \psi_f, \mathrm{id}_{B_\bullet}, \upsilon_z\}))$, a 3-morphism $\rho : \tau \Rrightarrow \upsilon$ is a pair:

$$(\{A_\bullet, B_\bullet, C_\bullet, D_\bullet\}, \{\phi_f, \psi_f, id_{B_\bullet}, \tau_z, \upsilon_z, 0, \Omega\})$$

such that $\Omega : A_\bullet \to D_{\bullet+2}$ satisfies $\Omega_{m-1} \circ d_m - d_{m+2} \circ \Omega_m = \tau_z - \upsilon_z$. This turns out to be the same as the definition of higher homotopy that one obtains by considering chain maps from the total complex $\mathrm{tot}(\mathrm{tot}(A_\bullet \otimes I_\bullet) \otimes I_\bullet)$ to $B_\bullet$.

From here it is easy to see the pattern: 3-morphisms $\rho, \pi : \tau \Rrightarrow \upsilon$ are 4-isomorphic if there exists a $\Xi : A_\bullet \to B_{\bullet+3}$ such that $\Xi_{m-1} \circ d_m + d_{m+3} \circ \Xi_m = \rho_\Omega - \pi_\Omega$, and so on. Informally, an n-morphism between two $(n-1)$ morphisms α, β is a degree $n-1$ map $\Phi : A_\bullet \to B_{\bullet+n-1}$ such that $\Phi_{m-1} \circ d_m + (-1)^n d_{m+n-1} \circ \Phi_m = \alpha - \beta$ (where we have neglected to specify the simplicial set structure).

Lastly, we describe the composition of morphisms. We can use the construction in Theorem 21 to obtain expressions for the 'horn fillers', i.e. extensions of horns Λ^n_k. In Section 3, we saw that composition corresponds precisely to such extensions, so we can write down expressions for these composites in the case of chain complexes.

Let $\phi = (\{A_\bullet, B_\bullet\}, \{\phi_f\})$ and $\psi = (\{B_\bullet, C_\bullet\}, \{\psi_f\})$. Applying the expression (2) from Theorem 21 when $n = 2$ and $k = 1$ gives the map $\psi_f \circ \phi_f$, and the extension is:

$$\tau := (\{A_\bullet, B_\bullet\, C_\bullet\}, \{\phi_f, \psi_f, \psi_f \circ \phi_f, 0\})$$

which is indeed a 2-simplex. The composite of 1-morphisms ϕ and ψ is then given by: $\partial_1(\tau) = (\{A_\bullet, C_\bullet\}, \{\psi_f \circ \phi_f\})$, as expected.

2-morphisms, as discussed in Section 3, can be composed three at a time, in two different configurations, corresponding to the two horns of Δ^3. A special case is vertical composition of pairs of 2-morphisms: if $\phi, \psi, \chi : A_\bullet \to B_\bullet$ with $\tau : \phi \Rightarrow \psi$ and $\upsilon : \psi \Rightarrow \chi$, then the construction in Theorem 21 yields the composite 2-morphism $\phi \Rightarrow \chi$ given by $(\{A_\bullet, B_\bullet\}, \{\phi_f, \chi_f, \mathrm{id}_{B\,\bullet}, \tau_z + \upsilon_z\})$. We can verify that indeed, $\tau_z + \upsilon_z$ is a homotopy from ϕ_f to χ_f.

Similarly for horizontal composition: if $\phi, \psi : A_\bullet \to B_\bullet$, $\pi, \chi : B_\bullet \to C_\bullet$, with $\tau : \phi \Rightarrow \psi$ and $\upsilon : \pi \Rightarrow \chi$, then there is a composite[4] given by $(\{A_\bullet, B_\bullet, C_\bullet\}, \{\phi_f, \pi_f, (\chi \circ \psi)_f, \upsilon_z \circ \psi_f + \pi_f \circ \tau_z\})$. Again, we can verify that $\upsilon_z \circ \psi_f + \pi_f \circ \tau_z$ is a homotopy from $\pi_f \circ \phi_f$ to $\chi_f \circ \psi_f$.

6 Conclusion

What we have seen is that all of the desired ∞-categorical structure of chain complexes arises as a purely combinatorial consequence of our chosen model. Working out the form of n-morphisms is routine, if laborious (and should lend itself well to manipulation by a computer). The should be contrasted with the "naïve approach" to ∞-category theory, attempted in the introduction, where nothing was routine.

The methods that we have used can be applied elsewhere to build ∞-categories. In particular, we can make an quasi-category out of any differential graded category, and the morphisms will not be hard to characterise, as we have seen.

[4]Note that in this case the composite 2-simplex we describe does not strictly fit our definition of 2-morphism, though it can be used to construct the desired 2-morphism.

References

[1] M. Boardman, and R. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, **347**, Springer-Verlag, 1973.

[2] A. Joyal, Quasi-categories and Kan complexes, *J. Pure Appl. Algebra* (2002), **17**(5):207-222.

[3] C. A. Weibel, *An Introduction to Homological Algebra*, Cambridge University Press, Cambridge (1994).

[4] J. Holstein, *Homological and Homotopical Algebra* [Lecture Notes] (2015), available at `www.srcf.ucam.org/~jvsh2/hha.html`.

[5] J. Lurie, *Higher Topos Theory*, eprint (2006) arXiv:math/0608040.

[6] J. Lurie, *Derived Algebraic Geometry I: Stable ∞-Categories*, eprint (2006) arXiv:math/0608228.

[7] J. Lurie, *Higher Algebra* (2016), available at `http://math.harvard.edu/~lurie/papers/HA.pdf`.